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ON ST. VENANT'S PRINCIPLE IN THE TORSION PROBLEM FOR A LAMINATED CYLINDER*

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The example of the torsion problem is used to show that, in a radially inhomogeneous cylinder with alternating hard and soft layers, weakly damped boundary layer solutions exist. The corresponding elementary solutions can penetrate quite deeply and essentially change the picture of the directionally deformed state remote from the end-faces. This in fact leads to violation of St. Venant's principle and of its classical statement. On the basis of an asymptotic analysis of the three-dimensional problem, a practical torsion theory is proposed, which adequately takes account of the singularities that arise. It was shown earlier /1/ that weakly damped boundary layer solutions exist for plates with alternating hard soft layers.

1. We consider the torsion problem for a circular radially inhomogeneous cylinder. The cylinder is given in cylindrical coordinates by

$$\Omega = \{r \in [r_0, r_1],\ \theta \in [0, 2\pi],\ z \in [0, L]\}$$

The equation of equilibrium is

$$\frac{\partial}{\partial \rho}\left[G\left(\frac{\partial u}{\partial \rho} - \frac{u}{\rho}\right)\right] + \frac{2}{\rho} G\left(\frac{\partial u}{\partial \rho} - \frac{u}{\rho}\right) + G\frac{\partial^2 u}{\partial \xi^2} = 0; \tag{1.1}$$
$$\rho = \frac{r}{r_1},\quad \xi = \frac{z}{r_1},\quad l = \frac{L}{r_1}$$

Here, u is the tangential component of the displacement vector, and $G = G(\rho)$ is the modulus of rigidity, which is regarded as a positive piecewise continuous function.

We assume that the lateral surface is free from stresses, and that boundary conditions are given on the end-faces.

The general solution of our problem can be written as

$$u = u_* + \sum_{s=1}^{\infty} v_s(\rho)\left[A_s e^{-\gamma_s \xi} + B_s e^{\gamma_s (\xi - l)}\right] \tag{1.2}$$
$$u_* = \rho\,(A_0 + B_0 \xi)$$

Here, u_* is the St. Venant solution, A_s, B_s are arbitrary constants, γ_s are positive eigenvalues, and v_s are the eigenfunctions of the spectral problem

$$\frac{d}{d\rho}\left[G\left(\frac{\partial v}{\partial \rho} - \frac{v}{\rho}\right)\right] + \frac{2}{\rho} G\left(\frac{\partial v}{\partial \rho} - \frac{v}{\rho}\right) + G\gamma^2 v = 0; \tag{1.3}$$
$$\frac{dv}{d\rho} - \frac{v}{\rho} = 0 \ \text{ for } \ \rho = \rho_0, \rho_1$$

It can be shown that problem (1.3) is selfadjoint, so that the eigenfunctions satisfy the orthogonality condition

$$\int_{\rho_1}^{\rho_s} \rho v_s v_t \, d\rho = \delta_{st}$$

where δ_{st} is the Kronecker delta. The constants A_s, B_s are found by satisfying the boundary conditions on the end-faces.

Some of the first numerical values of γ_s were quoted in /2/ for a homogeneous continuous and hollow cylinder, whence it follows that the exponential solutions have the nature of a boundary effect, localized on the end-faces, as is proved by St. Venant's principle. It will be shown below, by taking the example of a stratified cylinder, that this principle may be violated, because the individual eigenvalues γ_s may be quite small.

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2. Let the radially inhomogeneous cylinder consist of alternating hard and soft layers, numbering $n = 2r - 1$. We shall assume that the inner and outer layers are hard. Each hard layer is given the index $j = 1, 2, \ldots, r$, and each soft layer the index $i = 1, 2, \ldots, r-1$ (numbered from the centre). For simplicity we assume that all the hard and all the soft layers have the same elastic properties. The moduli of rigidity $G_j = G_h$; $G_i = G_s$. Let r_{1k} be the outer radius, and r_{0k} the inner radius, of the k-th layer; in dimensionless coordinates they are ρ_{1k} and ρ_{0k} respectively.

We introduce the parameter $p = G_s/G_h$ and examine the spectral problem (1.3) as $p \to 0$.

Theorem. The spectrum $\Lambda(p)$ of problem (1.3) is a denumerable real set and can be written as

$$\Lambda(p) = \Lambda_-(p) \cup \Lambda_+^{(1)}(p) \cup \Lambda_+^{(2)}(p) \tag{2.1}$$

1) $\Lambda_-(p)$ consists of the double eigenvalue $\gamma_0 = 0$ and of $2(r-1)$ eigenvalues of the type

$$\gamma_l = p^{1/2}\eta_l + O(p^{3/2})$$

where η_l are non-zero eigenvalues of the homogeneous Jacobian algebraic system (C is the Jacobi matrix)

$$Cx - \eta^2 Bx = 0 \tag{2.2}$$

$$x = (X_1, X_2, \ldots, X_r)^T, \quad B = \operatorname{diag} \| b_{jj} \|$$

$$C = \begin{Vmatrix} c_{11} & -c_{11} & 0 & 0 & \ldots & 0 & 0 \\ -c_{11} & c_{11}+c_{22} & -c_{22} & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & -c_{rr} & c_{rr} \end{Vmatrix}$$

$$b_{jj} = \frac{\rho_{1j}^4 - \rho_{0j}^4}{4}, \quad c_{jj} = \frac{2\rho_{1j}^2\rho_{0,j+1}^2}{\rho_{0,j+1}^2 - \rho_{1j}^2}$$

2) $\Lambda_+^{(1)}(p)$ consists of r sets of eigenvalues of the type

$$\gamma_{jt} = \gamma_{jt0} + O(p) \tag{2.3}$$

where γ_{jt0} is the root of the equation

$$L_{00}^{(j)} - \frac{2}{\gamma\rho_{1j}} L_{01}^{(j)} - \frac{2}{\gamma\rho_{0j}} L_{10}^{(j)} + \frac{4}{\gamma^2\rho_{0j}\rho_{1j}} L_{11}^{(j)} = 0 \tag{2.4}$$

3) $\Lambda_+^{(2)}(p)$ consists of $r-1$ sets of eigenvalues of the type

$$\gamma_{it} = \gamma_{it0} + O(p) \tag{2.5}$$

where γ_{it0} is the root of the equation

$$L_{11}^{(i)} = 0 \tag{2.6}$$

Here,

$$L_{\alpha\beta}^{(k)} = J_\alpha(\gamma\rho_{0k}) Y_\beta(\gamma\rho_{1k}) - J_\beta(\gamma\rho_{1k}) Y_\alpha(\gamma\rho_{0k})$$

and J_s, Y_s are Bessel functions of the first and second kind.

Since they are laborious, we shall only indicate the general scheme of the proofs of these results.

In our present case, the cylinder is piecewise-homogeneous in the radial direction, so that the spectral problem (1.3) reduces to a problem of conjugation, which in turn reduces to an algebraic system with a matrix whose elements depend analytically on the spectral parameter γ and linearly on the parameter p.

The above results are obtained by applying the perturbation theory of linear operators /3/ to the algebraic system mentioned. It is important to analyse the limit problem. Here, at $p = 0$, we have two limiting cases: 1) $G_s \to 0$, and the modulus of G_h is finite, 2) the modulus of G_s is finite, and $G_h \to \infty$.

Corresponding to the first case, we have a system of unconnected cylinders (hard layers), whose cylindrical surfaces are free from stress. The spectrum of the limiting problem is obviously the union of the sets of eigenvalues of the spectral problems that correspond to the individual cylinders. Denote these sets of $\Lambda_j(0)$. Each such set consists of the double eigenvalue $\gamma_{j0} = 0$ and a denumerable set of values γ_{jt0}, which are the first terms in (2.3) of the analytic expansions with respect to the parameter p. For small $p \neq 0$ the numbers γ_{j0} generate $\Lambda_-(p)$.

Corresponding to the second limiting case we have a system of $r-1$ unconnected cylinders on whose cylindrical surfaces the displacements are zero. Here, the spectrum of the limiting problem is again the union of $r-1$ sets of eigenvalues. Each such set $\Lambda_i(0)$ consists of the roots γ_{it0} of Eq. (2.6).

It follows from the theorem that the elementary solutions corresponding to $\Lambda_-(p)$ with small p are weakly damped on moving away from the end-faces, and can give a substantial correction to St. Venant's solution. We shall call the set of these solutions the weak boundary effect. The elementary solutions corresponding to $\Lambda_+^{(1)}$ and $\Lambda_+^{(2)}$ with small p, are rapidly (strongly) damped on moving away from the end-faces. We shall call the set of these the "strong boundary effect".

Let us give the expressions for the eigenfunctions v_l corresponding to $\Lambda_-(p)$, which describe the displacement distribution over the radius. We have (c_l is a normalizing factor)

$$v_l = c_l\,[v_{l0}(\rho) + O(p)] \tag{2.7}$$

$$v_{jl0} = X_{jl}\rho,\quad v_{il0} = (\rho_{1i}^2 - \rho_{0i}^2)^{-1}\times [X_{i+1,\,l}\rho_{1i}^2(\rho - \rho_{0i}^2/\rho) - X_{il}\rho_{0i}^2(\rho - \rho_{1i}^2/\rho)] \tag{2.8}$$

3. A simple mechanical interpretation can be given to St. Venant's solution together with the weak boundary effect.

In our radially inhomogeneous cylinder with alternating hard and soft layers, we shall assume that the cross-section $\xi = \mathrm{const}$ of a hard layer can only rotate about the cylinder axis without plane displacement. The displacements of points of the corss-section will then obviously have the form

$$u_j = g_j(\xi)\rho \tag{3.1}$$

We assume that the displacements in a soft layer in any section $\xi = \mathrm{const}$ as fully defined by the displacements of the adjacent hard layers, i.e., in accordance with (3.1) and our method of numbering, we have

$$u_i(\rho_{1i},\ \xi) = g_{i+1}(\xi)\rho_{1i};\ u_i(\rho_{0i},\ \xi) = g_i(\xi)\rho_{0i}$$

Under this hypothesis, we can write the displacements in the soft layer as

$$u_i = (\rho_{1i}^2 - \rho_{0i}^2)^{-1}[g_{i+1}\rho_{1i}^2(\rho - \rho_{0i}^2/\rho) - g_i\rho_{0i}^2(\rho - \rho_{1i}^2/\rho)] \tag{3.2}$$

In accordance with (3.1) and (3.2), the stress-strain state in each hard and soft layer is as follows:

$$\sigma_{\theta zj} = G_h e_{\theta zj},\quad e_{\theta zj} = \rho r_1^{-1} g_j' \tag{3.3}$$

$$\sigma_{r\theta i} = G_s e_{r\theta i},\quad e_{r\theta i} = \frac{2\rho_{0i}^2\rho_{1i}^2}{r_1\rho^2(\rho_{1i}^2 - \rho_{0i}^2)}(g_{i+1} - g_i)$$

The remaining components of the stress and strain tensors are zero.

To obtain the boundary value problem suitable for our model of the stress-strain state, we use the Lagrange variational principle

$$\delta\Pi - \delta A = 0 \tag{3.4}$$

where δA is the variation of the elementary work of the external forces, and $\delta\Pi$ is the variation of the strain energy. By (3.3), we have

$$\Pi = 2\pi r_1^3 \int_0^l \Big[\sum_{j=1}^{r}\int_{\rho_{0j}}^{\rho_{1j}} \sigma_{\theta zj} e_{\theta zj}\rho\, d\rho + \sum_{i=1}^{r-1}\int_{\rho_{0i}}^{\rho_{1i}} \sigma_{r\theta i} e_{r\theta i}\rho\, d\rho\Big]\, d\xi \tag{3.5}$$

To find δA, we assume e.g., that the following boundary conditions are given on the end-faces:

$$u_\theta(\rho, 0) = 0,\ \sigma_{z\theta}(\rho, l) = \tau(\rho) \tag{3.6}$$

$$\tau(\rho) = \begin{cases} \tau_j(\rho), & \rho \in [\rho_{0j}, \rho_{1j}] \\ 0, & \rho \in [\rho_{0i}, \rho_{1i}] \end{cases}$$

Regarding δg_j as independent variations ($\delta g_j = 0$ and $\xi = 0$), we obtain from the variational Eq.(3.4), using (3.3) and (3.6), a system of differential equations and boundary conditions which can be conveniently written in vector form (the matrices C and B are the same as in (2.2)):

$$-B\mathbf{g}'' + pC\mathbf{g} = 0 \tag{3.7}$$

$$\mathbf{g}(0) = 0,\ B\mathbf{g}'(l) = \mathbf{M}$$

$$\mathbf{g} = (g_1, g_2, \ldots, g_r)^T,\ \mathbf{M} = (M_1, M_2, \ldots, M_r) \tag{3.8}$$

$$M_j = 2\pi r_1^2 \int_{\rho_{0j}}^{\rho_{1j}} \tau_j\rho\, d\rho$$

If the solution of Eq.(3.7) is sought as

$$g = \mathbf{x}e^{\gamma\xi}, \quad \gamma = p^{1/2}\eta, \quad \mathbf{x} = (X_1, X_2, \ldots, X_r)^T$$

we arrive at problem (2.2).

Denote by $\mathbf{x}_t = (X_{1t}, X_{2t}, \ldots, X_{rt})^T$ the eigenvectors corresponding to the eigenvalue $\lambda_t = \eta_t^2$ of problem (2.2) $(t = 0, 1, \ldots, r-1)$. Since the problem is selfadjoint, they can be subjected to the condition

$$(B\mathbf{x}_t, \mathbf{x}_s) = \sum_{j=1}^{r} b_{jj} X_{jt} X_{js} = \delta_{st} \tag{3.9}$$

It can be shown directly that $\lambda_0 = 0$ is an eigenvalue, and the corresponding eigenvector is

$$\mathbf{x}_0 = (X_0, X_0, \ldots, X_0)^T, \quad X_0 = \Big(\sum_{j=1}^{r} b_{jj}\Big)^{-1/2} \tag{3.10}$$

The general solution of the vector equation can be written as

$$g = g_0 + \sum_{t=1}^{r-1} \mathbf{x}_t \left[A_t e^{-\gamma_t \xi} + B_t e^{\gamma_t(\xi - l)}\right] \tag{3.11}$$
$$g_0 = \mathbf{x}_0 (A_0 + B_0 \xi)$$

where A_0, B_0, A_t, B_t are arbitrary constants.

If the displacement field corresponding to the vector g_0 is constructed, we obtain on the basis of (3.1), (3.2) and (3.11):

$$u_i = X_0 (A_0 + B_0 \xi)\rho$$

i.e., this particular solution is exactly equivalent to the St. Venant's solution. The stress-strain state corresponding to the non-zero eigenvalue η_t is the first approximation with respect to p of the weak boundary effect. The displacement distribution over the radius, corresponding to $\gamma_t \neq 0$, is given by relations (2.8), which are in complete agreement with our hypotheses (3.1), (3.2).

We now find the constants A_0, B_0, A_t, B_t of the boundary conditions (3.8). Substituting (3.11) into (3.8) and using the orthogonality condition (3.9), we obtain

$$A_0 = 0, \quad B_0 = X_0 \sum_{j=1}^{r} M_j, \quad A_t = -e^{-\gamma_t l} B_t$$
$$B_t = \frac{m_t}{\gamma_t (1 - e^{-2\gamma_t l})}, \quad m_t = \sum_{j=1}^{r} M_j X_{jt}$$

4. Consider as an example a three-layer cylinder. Here,

$$\eta_1^2 = c_{11}(b_{11}/b_{22} + b_{22}/b_{11}), \quad X_0 = (b_{11} + b_{22})^{-1/2}$$
$$X_{11} = (b_{22}/b_{11})^{1/2} X_0, \quad X_{21} = -(b_{11}/b_{22})^{1/2} x_0$$

For the cylinder with parameters $\rho_{01} = 0, \rho_{11} = 0.5, \rho_{02} = 0.8, \rho_{12} = 1$, we have $\eta_1 = 7.62$. Hence, if $p = 10^{-2}$, then $\gamma_1 = 0.762$, and if $p = 10^{-3}$, then $\gamma_1 = 0.241$. According to /2/, for a continuous cylinder we have $\gamma_1 = 5.136$.

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